重點六 去零因子求極限

令 $P(x)$ 和 $Q(x)$ 皆為多項式，則：

1. $\lim_{x\to x_0} P(x) = P(x_0)$ 且 $\lim_{x\to x_0} Q(x) = Q(x_0)$

2. 若 $P(x_0) \neq 0$，則 $\lim_{x\to x_0} \dfrac{Q(x)}{P(x)} =$ __________

3. 若 $P(x_0) = 0$ 但 $Q(x_0) \neq 0$，則 $\lim_{x\to x_0} \dfrac{Q(x)}{P(x)} =$ __________

4. 看到 $P(x_0) = Q(x_0) = 0 \Rightarrow$ ____________________

5. 看到 分子代 x_0 = 分母代 $x_0 = 0 \Rightarrow$ ____________________

例題 1.
Find the following limits.

(1) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$ (2) $\lim_{x\to 2} \dfrac{x^4-16}{x-2}$ (3) $\lim_{x\to -1} \dfrac{x^3+1}{x^2-2x-3}$

解

例題 2. (精選範例 6-1)

Find the following limits.

(1) $\lim_{x\to 3}(\frac{1}{x}-\frac{1}{3})\frac{1}{x-3}$ (2) $\lim_{x\to 1}(\frac{x^3-1}{x^2-1}-\frac{x-\frac{1}{x}}{x-1})$

解

例題 3. (精選範例 6-2)

Find the following limits.

(1) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+5}-2}{x-1}$ (2) $\lim\limits_{x\to 2}\dfrac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$

解

重點六（補充）如何證明極限不存在

1. 若極限存在，譬如說 $\lim_{x\to x_0} f(x)=L$，
   則 $\forall \varepsilon>0, \exists \delta>0$ s.t. $\forall 0<|x-x_0|<\delta, |f(x)-L|<\varepsilon$

2. 所以若極限不存在，即對任意 $L\in\mathbb{R}$，$\lim_{x\to x_0} f(x)=L$ 均不成立
   針對任一個 $\lim_{x\to x_0} f(x)=L$，$\lim_{x\to x_0} f(x)=L$ 不成立的嚴格敘述如下：
   $\exists \varepsilon>0, \forall \delta>0$ s.t. $\exists x$ 滿足 $0<|x-x_0|<\delta, |f(x)-L|\geq\varepsilon$

3. 因此，$\lim_{x\to x_0} f(x)$ 不存在的定義爲：
   $\forall L\in\mathbb{R}, \exists \varepsilon>0, \forall \delta>0$ s.t. $\exists x$ 滿足 $0<|x-x_0|<\delta, |f(x)-L|\geq\varepsilon$

說例

(1) $\lim_{x\to 0}\frac{1}{x}$ 不存在

說明

1° Given $L\in\mathbb{R}$, W.L.O.G. may assume $L>0$, consider $\varepsilon=1$

Given $\delta>0$, choose $x=\min\{\frac{\delta}{2},\frac{1}{2(L+1)}\}$

$\because$ $\frac{\delta}{2}>0$ and $\frac{1}{2(L+1)}>0$

$\therefore$ $x>0$

$\Rightarrow$ $|x-0|=|x|=x>0$ and $|x-0|=|x|=x\leq\frac{\delta}{2}<\delta$

$\Rightarrow$ $0<|x-0|<\delta$

2° Now, for such x

If $\frac{\delta}{2}<\frac{1}{2(L+1)}$

then $x=\min\{\frac{\delta}{2},\frac{1}{2(L+1)}\}=\frac{\delta}{2}$ and $\frac{2}{\delta}>2(L+1)=2L+2$

It implies that $|f(x)-L|=|\frac{1}{x}-L|=|\frac{2}{\delta}-L|=|L+2|>1=\varepsilon$

If $\frac{\delta}{2}\geq\frac{1}{2(L+1)}$

then $x=\min\{\frac{\delta}{2},\frac{1}{2(L+1)}\}=\frac{1}{2(L+1)}=\frac{1}{2L+2}$

In this case, $|f(x)-L|=|\frac{1}{x}-L|=|(2L+2)-L|=|L+2|>1=\varepsilon$

3° Since L is arbitrary, $\lim_{x\to 0}\frac{1}{x}$ does not exist. [Q.E.D.]

(2) 設 $f(x)=\begin{cases}-1 & \text{if } x\in\mathbb{Q}\\ 1 & \text{if } x\notin\mathbb{Q}\end{cases}$，則 $\lim_{x\to 0}f(x)$ 不存在

說明

1° For $L\neq\pm 1$, choose $\varepsilon=\min\{\frac{|L-1|}{2},\frac{|L+1|}{2}\}$

Given $\delta>0$, choose $x=\frac{\delta}{2}>0$

$\Rightarrow$ $|x-0|=|x|=x>0$ and $|x-0|=|x|=x=\frac{\delta}{2}<\delta$

$\Rightarrow$ $0<|x-0|<\delta$

In this case, for such x

If $x\in\mathbb{Q}$, $|f(x)-L|=|-1-L|=|L+1|>\frac{|L+1|}{2}\geq\min\{\frac{|L-1|}{2},\frac{|L+1|}{2}\}=\varepsilon$

If $x\notin\mathbb{Q}$, $|f(x)-L|=|1-L|=|L-1|>\frac{|L-1|}{2}\geq\min\{\frac{|L-1|}{2},\frac{|L+1|}{2}\}=\varepsilon$

This show that $\lim_{x\to 0}f(x)\neq L$ for any $L\neq\pm 1$

2° For , choose $\varepsilon=1$

Given $\delta>0$, choose $x\in\mathbb{Q}$ with $0<|x-0|<\delta$

In this case, for such x

$|f(x)-L|=|-1-1|=2>1=\varepsilon$

This shows that $\lim_{x\to 0}f(x)\neq 1$

3° For $L=-1$, choose $\varepsilon=1$

Given $\delta>0$, choose $x\notin\mathbb{Q}$ with $0<|x-0|<\delta$

In this case, for such

$|f(x)-L|=|1-(-1)|=2>1=\varepsilon$

This shows that $\lim_{x\to 0}f(x)\neq -1$

4° By 1°, 2° and 3°, $\lim_{x\to 0}f(x)$ does not exist. [Q.E.D.]