重點四 極限運算定理 (四則運算篇)

◎ 設 $\lim\limits_{x\to x_0} f(x)=L$，$\lim\limits_{x\to x_0} g(x)=M$ 且 $c\in\mathbb{R}$

(1) $\lim\limits_{x\to x_0}[cf(x)]=$ ________

(2) $\lim\limits_{x\to x_0}[f(x)+g(x)]=$ ________

(3) $\lim\limits_{x\to x_0}[f(x)\cdot g(x)]=$ ________

(4) 若 $\boldsymbol{M\neq 0}$，則 $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=$ ________

說明

(1)

(2)

(3) 1° Given $\varepsilon > 0$, since $\lim_{x\to x_0} f(x) = L$,

$\exists \delta_1 > 0$ such that, $\forall 0 < |x - x_0| < \delta_1$, $|f(x) - L| < \dfrac{\varepsilon}{2|M|+1}$

$\exists \delta_2 > 0$, $\exists K > 0$ such that, $\forall 0 < |x - x_0| < \delta_2$, $|f(x)| < K$

$\because$ $\lim_{x\to x_0} g(x) = M$

$\therefore$ $\exists \delta_3 > 0$ such that, $\forall 0 < |x - x_0| < \delta_3$, $|g(x) - M| < \dfrac{\varepsilon}{2K}$

2° Let $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$,

then, $\forall 0 < |x - x_0| < \delta$, we have

$$\begin{aligned}|f(x)g(x) - LM| &\le |[f(x)g(x) - f(x)M] - [f(x)M - LM]| \\ &\le |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &\le |f(x)||g(x) - M| + |M||f(x) - L| \\ &< K \cdot \frac{\varepsilon}{2K} + |M| \cdot \frac{\varepsilon}{2|M|+1} < \varepsilon.\end{aligned}$$

3° Since ε is arbitrary, $\lim_{x\to x_0}[f(x)\cdot g(x)] = L \cdot M$. [Q.E.D.]

(4) 1° First we show that $\lim_{x\to x_0} \dfrac{1}{g(x)} = \dfrac{1}{M}$:

Given $\varepsilon > 0$, since $\lim_{x\to x_0} g(x) = M$,

$\exists \delta_1 > 0$ such that, $\forall 0 < |x - x_0| < \delta_1$, $|g(x) - M| < 2\varepsilon$

$\exists \delta_2 > 0$ such that, $\forall 0 < |x - x_0| < \delta_2$, $|g(x) - M| < \dfrac{|M|}{2}$

So, $\forall 0 < |x - x_0| < \delta_2$, we have

$|g(x)| = |[g(x) - M] + M| \ge |M| - |g(x) - M| > |M| - \dfrac{|M|}{2} = \dfrac{|M|}{2}$,

or equivalently, $\dfrac{1}{|g(x)|} < \dfrac{2}{|M|}$

2° Let $\delta = \min\{\delta_1, \delta_2\} > 0$,

then, $\forall 0 < |x - x_0| < \delta$, we have

$$\left|\frac{1}{g(x)}-\frac{1}{M}\right|=\left|\frac{M-g(x)}{g(x)M}\right|=\frac{|g(x)-M|}{|g(x)||M|}<\frac{2\varepsilon}{\frac{2}{|M|}\cdot|M|}=\varepsilon$$

3° Since ε is arbitrary, $\lim_{x\to x_0}\frac{1}{g(x)}=\frac{1}{M}$.

4° By (3), we see that $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}[f(x)\cdot\frac{1}{g(x)}]=L\cdot\frac{1}{M}=\frac{L}{M}$. [Q.E.D.]

例題 1.
Find the following limits.

(1) $\lim_{x\to 1}(x^2+2x-1)$　(2) $\lim_{x\to 1}\frac{2x-5}{3x+2}$　(3) $\lim_{x\to -\frac{2}{3}}\frac{2x-5}{3x+2}$

解

例題 2. (精選範例 4-1)

Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, show that $\lim_{x\to x_0}P(x)=P(x_0)$

解

例題 3. (精選範例 4-2)

Show that $\lim\limits_{x \to x_0} x^{\frac{n}{m}} = x_0^{\frac{n}{m}}$, where $m, n \in \mathbb{N}$. ($x_0 > 0$)

解

例題 4. (精選範例 4-3)

Show that

(1) $\lim\limits_{x \to x_0} \tan x = \tan x_0$ for $x_0 \neq \dfrac{2k+1}{2}\pi$, $k \in \mathbb{Z}$

(2) $\lim\limits_{x \to x_0} \sec x = \sec x_0$ for $x_0 \neq \dfrac{2k+1}{2}\pi$, $k \in \mathbb{Z}$

解

重點四（補充）極限的局部有界性質

1. 若一函數 $f(x)$ 在 $x = x_0$ 的極限存在，
   則 $\exists\delta > 0$、$M > 0$ 使得 $\forall 0 < |x - x_0| < \delta$ 均有 $|f(x)| < M$

   說明

   1° Suppose that $\lim\limits_{x\to x_0} f(x) = L$.

   $\Rightarrow$ $\exists\delta > 0$ such that, $\forall 0 < |x - x_0| < \delta$, $|f(x) - L| < 1$

   $\Rightarrow$ $\forall 0 < |x - x_0| < \delta$, $L - 1 < f(x) < L + 1$

   2° Let $M = \max\{|L+1|, |L-1|\}$

   $\because \begin{cases} L + 1 \le |L+1| \le M \\ L - 1 \ge -|L-1| \ge -M \end{cases}$

   $\therefore$ $\forall 0 < |x - x_0| < \delta$, $-M \le L - 1 < f(x) < L + 1 \le M$

   $\Rightarrow$ $\forall 0 < |x - x_0| < \delta$, $|f(x)| < M$ [Q.E.D.]