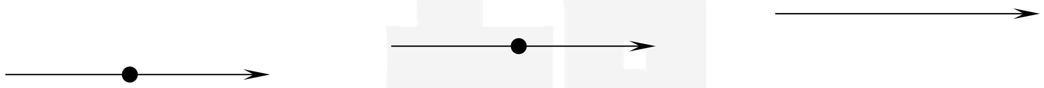
重點三 微分合成律 (連鎖律)

若 $f(x)$ 在 $x = x_0$ 可微，且 $g(x)$ 在 $x = f(x_0)$ 可微，

則 $(g \circ f)(x)$ 在 $x = x_0$ 必可微且 $(g \circ f)'(x_0) =$ ______________



說明

$1°$ Define $F(y) = \begin{cases} \dfrac{g(y) - g(f(x_0))}{y - f(x_0)} & \text{if } y \neq f(x_0) \\ g'(f(x_0)) & \text{if } y = f(x_0) \end{cases}$

$$\lim_{y \to f(x_0)} F(y) = \lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)} = g'(f(x_0)) = F(f(x_0))$$

$2°$ For $t \neq x_0$,

if $f(t) = f(x_0)$,

then $\dfrac{g(f(t)) - g(f(x_0))}{t - x_0} = 0 = F(f(t)) \cdot \dfrac{f(t) - f(x_0)}{t - x_0}$;

If $f(t) \neq f(x_0)$,

then $\dfrac{g(f(t)) - g(f(x_0))}{t - x_0} = \dfrac{g(f(t)) - g(f(x_0))}{f(t) - f(x_0)} \cdot \dfrac{f(t) - f(x_0)}{t - x_0} = F(f(t)) \cdot \dfrac{f(t) - f(x_0)}{t - x_0}$.

$3°$ So, $(g \circ f)'(x_0) = \lim\limits_{t \to x_0} \dfrac{(g \circ f)(t) - (g \circ f)(x_0)}{t - x_0} = \lim\limits_{t \to x_0} \dfrac{g(f(t)) - g(f(x_0))}{t - x_0}$

$= \lim\limits_{t \to x_0} [F(f(t)) \cdot \dfrac{f(t) - f(x_0)}{t - x_0}] = F(f(x_0)) f'(x_0) = g'(f(x_0)) f'(x_0)$ [Q.E.D.]

例題 1.

Calculate (1) $[(2x+5)^{100}]'$ (2) $[\sin(2x+5)]'$ (3) $(2^{\sin x})'$ (4) $[\log_3(5^x+1)]'$

解

例題 2. (精選範例 3-1)

Show that $|x|' = \dfrac{x}{|x|}$ and $|f(x)|' = \dfrac{f(x)}{|f(x)|} f'(x)$.

解

例題 3. (精選範例 3-2)

Let $f(x) = x^p$, where $p \in \mathbb{Q}$. Show that $f'(x) = px^{p-1}$.

解

例題 4. (精選範例 3-3)

Differentiate the following functions.

(1) $f(x)=|2x+5|$ (2) $f(x)=|\sin x|+|\cos x|$ (3) $f(x)=(\sin x)^{\frac{3}{2}}$

(4) $f(x)=\log_2|x+1|$ (5) $f(x)=\left|\dfrac{\sin x}{x}\right|$

解