重點二 微分的運算律

設 $f(x)$ 和 $g(x)$ 均在 $x=x_0$ 可微，則：

(1) $(c\cdot f)(x)$ 在 $x=x_0$ 必可微且 $(c\cdot f)'(x_0)=$ ________

(2) $(f+g)(x)$ 在 $x=x_0$ 必可微且 $(f+g)'(x_0)=$ ________

(3) $(f\cdot g)(x)$ 在 $x=x_0$ 必可微且 $(f\cdot g)'(x_0)=$ ________________

(4) **若 $\mathbf{g(x_0)\neq 0}$**，則 $(\frac{f}{g})(x)$ 必可微且 $(\frac{f}{g})'(x_0)=$ ________________

說明

(1) $(c\cdot f)'(x_0)=$

(2) $(f+g)'(x_0)=$

(3) $$(f+g)'(x_0)=\lim_{h\to 0}\frac{(f\cdot g)(x_0+h)-(f\cdot g)(x_0)}{h}$$
$$=\lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h}$$
$$=\lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0+h)+f(x_0)g(x_0+h)-f(x_0)g(x_0)}{h}$$
$$=\lim_{h\to 0}[\frac{f(x_0+h)-f(x_0)}{h}\cdot g(x_0+h)+f(x_0)\cdot\frac{g(x_0+h)-g(x_0)}{h}]$$
$$=f'(x_0)g(x_0)+f(x_0)g'(x_0)$$ [Q.E.D.]

(4) $$\because\ (\frac{1}{g})'(x_0)=\lim_{h\to 0}\frac{(\frac{1}{g})(x_0+h)-(\frac{1}{g})(x_0)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{g(x_0+h)}-\frac{1}{g(x_0)}}{h}$$
$$=\lim_{h\to 0}\frac{\frac{g(x_0)-g(x_0+h)}{g(x_0+h)g(x_0)}}{h}$$
$$=\lim_{h\to 0}\frac{-\frac{g(x_0+h)-g(x_0)}{h}}{g(x_0+h)g(x_0)}$$
$$=\frac{-g'(x_0)}{(g(x_0))^2}$$

$$\therefore\ (\frac{f}{g})'(x_0)=(f\cdot\frac{1}{g})'(x_0)$$
$$=f'(x_0)(\frac{1}{g})(x_0)+f(x_0)(\frac{1}{g})'(x_0)$$
$$=\frac{f'(x_0)}{g(x_0)}+f(x_0)\cdot\frac{-g'(x_0)}{(g(x_0))^2}$$
$$=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}$$ [Q.E.D.]

口訣

◎ 相乘的微分 ⇒ ________________

◎ 相除的微分 ⇒ ________________

例題 1.

Show that $(\tan x)' = \sec^2 x$ and $(\sec x)' = \sec x \tan x$.

解

例題 2. (精選範例 2-1)

Find the derivative of the given function $f(x)$ at the point $x = x_0$.

(1) $f(x) = 2x^3 + 5$, $x_0 -1$

(2) $f(x) = \dfrac{4x+3}{2x-1}$, $x_0 = 1$

(3) $f(x) = \sin x \cos x$, $x_0 = 0$

(4) $f(x) = a^x \log_a x$, $x_0 = 1$

解