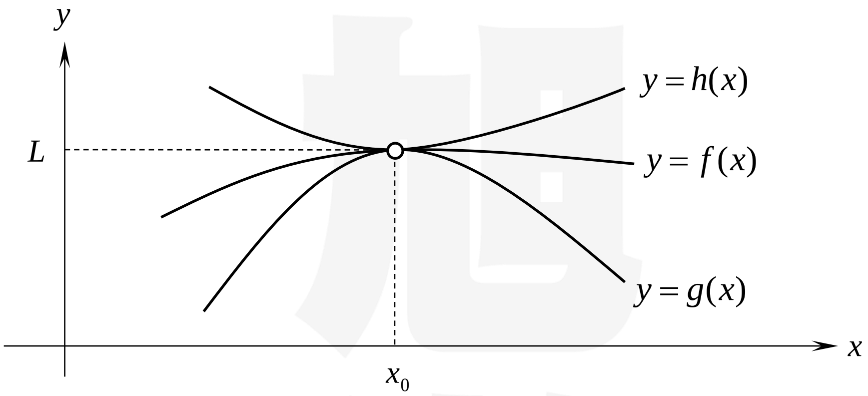
重點十一 夾擠定理

1. 定理內容：

若 $\begin{cases} \exists \eta > 0 \text{ 使得 } \forall 0 < |x - x_0| < \eta \text{ 皆有 } g(x) \le f(x) \le h(x) \\ \lim\limits_{x \to x_0} g(x) = \lim\limits_{x \to x_0} h(x) = L \end{cases}$，則：________________



說明

1° Given $\varepsilon > 0$, since $\lim\limits_{x \to x_0} g(x) = \lim\limits_{x \to x_0} h(x) = L$,

$\exists \delta_0 > 0$ such that, $\forall 0 < |x - x_0| < \delta_0$, $|g(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$.

So, $\forall 0 < |x - x_0| < \delta_0$, $g(x) > L - \varepsilon$ and $h(x) < L + \varepsilon$

2° Let $\delta = \min\{\eta, \delta_0\}$,

then, $\forall 0 < |x - x_0| < \delta$, we have

$L - \varepsilon < g(x) \le f(x) \le h(x) < L + \varepsilon$

$\Rightarrow \; -\varepsilon < f(x) - L < \varepsilon$

$\Rightarrow \; |f(x) - L| < \varepsilon$

3° Since ε is arbitrary, $\lim\limits_{x \to x_0} f(x) = L$. [Q.E.D.]

2. 此定理當 $x_0 = \infty$ 或 $x_0 = -\infty$ 時也可以使用！

例題 1.

Show that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

解

例題 2.

Find the following limits.

(1) $\lim_{x\to\infty}\frac{\sin x}{x}$ (2) $\lim_{x\to 0}(x\sin\frac{1}{x})$

解

例題 3. (精選範例 11-1)

Let $f(x)=\begin{cases} x^2, & \text{if } x\in\mathbb{Q} \\ 0, & \text{if } x\notin\mathbb{Q} \end{cases}$ and $g(x)=\begin{cases} x, & \text{if } x\in\mathbb{Q} \\ 0, & \text{if } x\notin\mathbb{Q} \end{cases}$, show that

(1) $\lim\limits_{x\to 0} f(x)=0$ (2) $\lim\limits_{x\to 0} g(x)=0$

解

例題 4. (精選範例 11-2)

Find the following limits.

(1) $\lim_{x\to\infty} 3^{\frac{1}{x}}$ (2) $\lim_{x\to\infty} \sqrt[x]{3^x+5^x+7^x}$

解