重點四 積分運算性質

設 $f(x)$ 和 $g(x)$ 都是在 $[a,b]$ 上可積的函數，且 $c \in \mathbb{R}$ 爲一常數。

1. 四則運算篇：

(1) $\int_a^b c \cdot f(x)dx = $ _____

(2) $\int_a^b f(x) + g(x)dx = $ _____

(3) $\int_a^b f(x) \cdot g(x)dx = $ _____

(4) $\int_a^b \dfrac{f(x)}{g(x)}dx = $ _____

說明

(1) Let P be a partition of $[a,b]$,

$\because$ $f(x)$ is continuous on $[a,b]$

$\therefore$ $f(x)$ is integrable on $[a,b]$

$\Rightarrow \lim\limits_{\|P\| \to 0} U_{f,[a,b],P} = \lim\limits_{\|P\| \to 0} L_{f,[a,b],P} = A = \int_a^b f(x)dx \in \mathbb{R}$

$\because \lim\limits_{\|P\| \to 0} U_{cf,[a,b],P} = \lim\limits_{\|P\| \to 0} \sum cM_k \Delta x_k = c \lim\limits_{\|P\| \to 0} \sum M_k \Delta x_k = c \lim\limits_{\|P\| \to 0} U_{f,[a,b],P} = cA$

and $\lim\limits_{\|P\| \to 0} L_{cf,[a,b],P} = \lim\limits_{\|P\| \to 0} \sum cm_k \Delta x_k = c \lim\limits_{\|P\| \to 0} \sum m_k \Delta x_k = c \lim\limits_{\|P\| \to 0} L_{f,[a,b],P} = cA$

$\therefore \int_a^b c \cdot f(x)dx = cA = c\int_a^b f(x)dx$

(2) $\because \lim\limits_{\|P\| \to 0} U_{f+g,[a,b],P} = $

and $\lim\limits_{\|P\| \to 0} L_{f+g,[a,b],P} = $
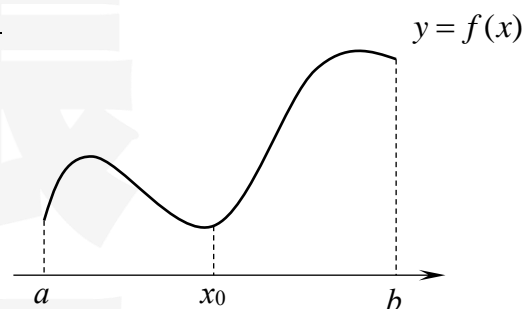
$\therefore \int_a^b f(x) + g(x)dx = $

2. 看圖說**等式**篇： (此處 $x_0 \in [a,b]$)

    (1) $\int_a^b f(x)dx = \int_a^{x_0} f(x)dx + $ _____

    (2) $\int_{x_0}^{x_0} f(x)dx = $ _____

    (3) $\int_a^b f(x)dx = $ _____
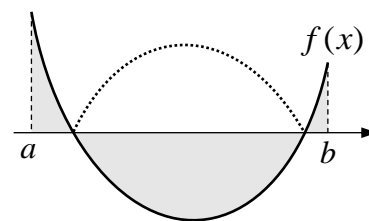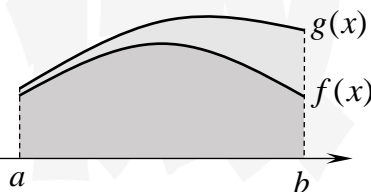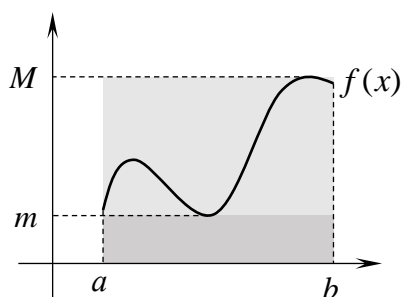
$y = f(x)$

a x_0 b

3. 看圖說**不等式**篇：

    (1)   若 $m \le f(x) \le M$ ，則 _____ $\le \int_a^b f(x)dx \le$ _____

    (2)   若 $f(x) \le g(x)$ ，則 $\int_a^b f(x)dx$ _____ $\int_a^b g(x)dx$

    (3)   若 $a \le b$ ，則 $\left| \int_a^b f(x)dx \right|$ _____ $\int_a^b |f(x)|dx$

M $f(x)$

m

a b

$g(x)$

$f(x)$

a b

$f(x)$

a b

例題 1.

(1) $\int_a^b 2x\,dx = ?$ (2) $\int_a^b 2x + 5\,dx = ?$

解

張旭老師 編授

例題 2. (精選範例 4-1)

Suppose that $\int_1^3 f(x)dx = 5$, $\int_2^5 f(x)dx = 7$, $\int_2^3 f(x)dx = 2$, and $\int_4^5 f(x)dx = -1$.

Evaluate the following integrals:

(1) $\int_1^5 f(x)dx$ (2) $\int_4^2 f(x)dx$ (3) $\int_2^5 f(x)dx$

解

例題 3. (精選範例 4-2)

Show that $\int_0^1 x^n dx \le \dfrac{1}{2}$ for all $n \in \mathbb{N}$.

解

例題 4. (精選範例 4-3)

Let $f(x)$ be continuous on $[a,b]$, show that there is a number $c \in [a,b]$ such that

$\int_a^b f(x)dx = f(c) \cdot (b-a)$.

解