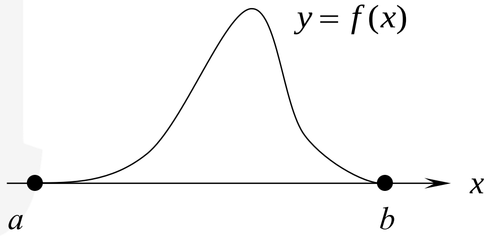
第四章 微分的應用

- 只有 exp(x) 能讓我微所欲微

重點一 均值定理

1. 洛爾均值定理：

設 $f(x)$ 爲一個在 $[a,b]$ 上連續且在 (a,b) 上可微的函數，
若 $f(a)=f(b)=0$，
則 ______________________________



說明

1° If $f(x)$ is constant on $[a,b]$, then done.

2° Suppose $f(x)$ is not constant on $[a,b]$.
W.L.O.G., may assume that $f(x)>0$ for some $x\in(a,b)$.

3° $\because$ $f(x)$ is continuous on $[a,b]$
$\therefore$ $\exists c\in[a,b]$ such that $f(c)\ge f(x)$ for all $x\in[a,b]$
$\because$ $f(x)>0$ for some $x\in(a,b)$
$\therefore$ $f(c)>0$
$\Rightarrow$ $c\in(a,b)$ and $f'(c)$ exists

4° Finally, since $f(c)$ is the maximum and $f'(c)$ exists,
we have $f'(c)=0$ [Q.E.D.]

2. 均值定理：

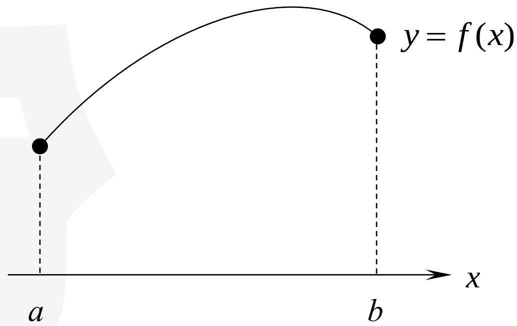
設 $f(x)$ 爲一個在 $[a,b]$ 上連續且在 (a,b) 上可微的函數，
則存在 $c\in(a,b)$ 使得 [　　　　　　　　　　]

說明

Let $g(x)=f(x)-L(x)$,

where $L(x)=f(a)+\dfrac{f(a)-f(b)}{a-b}(x-a)$,

hen $\begin{cases} ① \ g(a)=g(b)=0 \\ ② \ g(x) \text{ is continuous on } [a,b] \\ ③ \ g(x) \text{ is differentiable on } (a,b) \end{cases}$



So, by Rolle's theorem,

$\exists c \in (a,b)$ such that $g'(c)=0$

$\Rightarrow \quad f'(c)-L'(c)=0$

$\Rightarrow \quad f'(c)=L'(c)=\dfrac{f(a)-f(b)}{a-b}$ [Q.E.D.]

例題 1.
Let $f(x)=x^4-2x^2-8$, show that there exists $c\in[-2,2]$ such that $f'(c)=0$.
解

例題 2. (精選範例 1-1)
Suppose that $f(x)$ is differentaible on $(2,6)$ and continuous on $[2,6]$. Given that $1\le f'(x)\le 3$ for all x in $(2,6)$, show that $4\le f(6)-f(2)\le 12$.
解

例題 3. (精選範例 1-2)

Show that

(1) $|\sin x - \sin y| \le |x - y|$ for any $x, y \in \mathbb{R}$

(2) $|\sin x| \le |x|$ for any $x \in \mathbb{R}$

解

例題 4. (精選範例 1-3)

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a nonconstant polynomial. Show that between any two consecutive roots of the equation $P'(x) = 0$ there is at most one root of the equation $P(x) = 0$.

解

例題 5. (精選範例 1-4)

Prove that if $f(x)$ is differentiable on an interval I and $f'(x)<1$ for all $x\in I$, then there is at most one $c\in I$ such that $f(c)=c$.

解

例題 6. (精選範例 1-5) (Cauchy's mean-value theorem)

Suppose that $f(x)$ and $g(x)$ both satisfy the hypothesis of the mean-value theorem. Prove that if $g'(x)\neq 0$ far all $x\in(a,b)$, then there exists at least one number $c\in(a,b)$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(c)}{g'(c)}$.

解