重點二 微分與極限的聯手 (羅必達法則)

1. 羅必達法則：

設 $f(x)$ 和 $g(x)$ 在 $[a,b]$ 上連續且在 (a,b) 上可微。

若 $\begin{cases} ① \lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0 \\ ② \text{對任意 } x\in(a,b),\ x\neq x_0,\ \text{均有 } g'(x)\neq 0 \end{cases}$

則 $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} =$

說明

1° ∵ $f(x)$ and $g(x)$ are both differentiable at $x = x_0$

∴ $f(x)$ and $g(x)$ are both continuous at $x = x_0$

$\Rightarrow f(x_0) = \lim_{x\to x_0} f(x) = 0$ and $g(x_0) = \lim_{x\to x_0} g(x) = 0$

2° For any $t \in (a, x_0)$,

since $\begin{cases} ① f(x) \text{ and } g(x) \text{ are continuous on } [t, x_0] \\ ② f(x) \text{ and } g(x) \text{ are differentiable on } (t, x_0] \end{cases}$

by Cauchy's mean-value theorem,

there exist $\xi \in (t, x_0)$ such that $\dfrac{f'(\xi)}{g'(\xi)} = \dfrac{f(t)-f(x_0)}{g(t)-g(x_0)} = \dfrac{f(t)}{g(t)}$

3° Letting $t \to x_0^-$,

then $\xi \to x_0^-$ and thus $\lim_{t\to x_0^-} \dfrac{f(t)}{g(t)} = \lim_{t\to x_0^-} \dfrac{f'(\xi)}{g'(\xi)} = \lim_{t\to x_0^-} \dfrac{f'(t)}{g'(t)}$.

Thus $\lim_{x\to x_0^-} \dfrac{f(x)}{g(x)} = \lim_{x\to x_0^-} \dfrac{f'(x)}{g'(x)}$.

4° Similarly, $\lim_{x\to x_0^+} \dfrac{f(x)}{g(x)} = \lim_{x\to x_0^+} \dfrac{f'(x)}{g'(x)}$.

5° By 3° and 4°, we see that $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \lim_{x\to x_0} \dfrac{f'(x)}{g'(x)}$. [Q.E.D.]

2. 注意事項：

(1) 羅必達法則當 $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \infty$ 時也可以使用！

(2) 羅必達法則當 $x_0 = \infty$ 或 $x_0 = -\infty$ 時也可以使用！

(3) 若使用羅必達法則以後還是 $\frac{0}{0}$ 或 $\frac{\infty}{\infty}$ 不定型，可再使用一次羅必達法則！

例題 1.

Use L'Hopital's rule to find the following limits.

(1) $\lim_{x\to 0}\frac{\sin x}{x}$ (2) $\lim_{x\to \infty}\frac{\sin x}{x}$ (3) $\lim_{x\to 0}\frac{1-\cos x}{x}$ (4) $\lim_{x\to \infty}\frac{1-\cos x}{x}$ (5) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

解

例題 2.

Find the following limits.

(1) $\lim_{x\to 0}\frac{\tan x - x}{x^3}$ (2) $\lim_{x\to 0}\frac{x-\tan^{-1}x}{x^3}$ (3) $\lim_{x\to 0}\frac{\sin^{-1}x - x}{x^3}$ (4) $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$

解

例題 3.

Let $f(x)=\begin{cases}\dfrac{1-\cos x}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$. Find $f'(0)$ and $f''(0)$.

解